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DONOVAN'S CONJECTURE, CROSSED PRODUCTS AND ALGEBRAIC GROUP ACTIONS

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ABSTRACT

Donovan's conjecture, on blocks of finite group algebras over an algebraically closed field of prime characteristic p, asserts that for any finite p-group D, there are only finitely many Morita equivalence classes of blocks with defect group D. The main result of this paper is a reduction theorem: It suffices to prove the conjecture for groups generated by conjugates of D. A number of other finiteness results are proved along the way. The main tool is a result on actions of algebraic groups.

1. Introduction

One of the major unsolved problems in modular representation theory of finite groups is the following one.

DONOVAN'S CONJECTURE: *For any prime p and any finite p-group D, there are only finitely many Morita equivalence classes of p-blocks of finite groups with defect groups isomorphic to D.*

The conjecture is known to be true when D is cyclic [2, 6] and (up to minor ambiguities) when $p = 2$ and D is dihedral, semi-dihedral or quaternion [5]. It is also known to be true when one restricts attention to p-blocks of p-solvable groups only [8]. The case of blocks of symmetric groups is dealt with in [13].

In this paper we reduce the general case of Donovan's conjecture to the special case of blocks with defect group D, in finite groups generated by conjugates of

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D. A number of other results (which we consider to be interesting in their own right) are proved along the way. A brief outline is as follows.

We use Dade's theory of block extensions [4] to write every p-block with defect group D (up to Morita equivalence) as a crossed product Y of a finite group X (whose order is not divisible by p and bounded in terms of D) with an algebra R which is the basic subalgebra of a p-block with defect group D, in a finite group generated by conjugates of D. We then prove a finiteness theorem for such crossed products. This finiteness theorem is in turn based on two other finiteness results. One of these shows that, for any finite group G and any commutative G-algebra C over an algebraically closed field F whose characteristic p does not divide the order of G, the cohomology group $\mathrm{H}^i(G, \mathrm{U}(C))$ is finite for $i \geq 1$. The other states that, for any finite group G whose order is not divisible by p and any linear algebraic group H over F, there are only finitely many homomorphisms $G \to H$, up to conjugation within H. These results together give the proof of our reduction theorem.

For the rest of this paper, we denote by F an algebraically closed field of characteristic p (where we allow the case $p = 0$ for a while).

2. Representations of finite groups in algebraic groups

In this section we deal with homomorphisms from a finite group G into a linear algebraic group H over F. We refer to these homomorphisms as **representations** of G in H. Examples are the **linear** representations ($H = \mathrm{GL}(n, F)$ for some n), **projective** representations ($H = \mathrm{PGL}(n, F)$ for some n), **orthogonal** representations ($H = \mathrm{O}(n, F)$ for some n), and **symplectic** representations ($H = \mathrm{Sp}(2n, F)$ for some n). Further examples are provided by **G-algebras** ($H = \mathrm{Aut}(A)$, the automorphism group of a finite-dimensional algebra A over F) and by **interior G-algebras** ($H = \mathrm{U}(A)$, the group of units of a finite-dimensional algebra A over F). Later in this paper we will consider the example where $H = \mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$, the outer automorphism group of a finite-dimensional algebra A over F.

Two representations ρ, σ of a finite group G in a linear algebraic group H over F are called **equivalent** if there is an element $h \in H$ such that $\sigma(g) = h\rho(g)h^{-1}$ for $g \in G$. We have the following finiteness result.

THEOREM: *Let H be a linear algebraic group over an algebraically closed field F of characteristic p, and let G be a finite group whose order is not divisible* *by p (e.g. $p = 0$). Then there are only finitely many equivalence classes of representations of G in H.*

The case where $H = \mathrm{GL}(n, F)$ is an immediate consequence of Maschke's theorem. The general case was conjectured in an unpublished preprint by the author [11]. In that preprint it was also shown that in a "minimal" counterexample to the theorem G would be nonabelian simple and H would be simple of one of the exceptional types E_6, E_7, E_8 or F_4. In particular, the result was proved for solvable G and hence, by the Odd Order Theorem, for the case when $p = 2$.

As a response to [11], the author received proofs of the theorem above by T.A. Springer and S. Donkin. Moreover, A. Borel pointed out that the result follows without difficulty from a paper by A. Weil [15]. These proofs make use of the fact that the set of all representations of G in H can be considered as an affine algebraic variety over F on which the linear algebraic group H acts morphically.

We would like to take the opportunity to ask (as in [11]) whether the theorem above can be extended in the following way.

QUESTION: *Let F be an algebraically closed field of prime characteristic p, let H be a linear algebraic group over F, let G be a finite group, let P be a Sylow p-subgroup of G, and let ρ be a representation of P in H. Then, are there only finitely many equivalence classes of representations of G in H which contain a representation extending ρ?*

This is true for the special case $H = \mathrm{GL}(n, F)$ since every FG-module M is isomorphic to a direct summand of $\mathrm{Ind}_P^G(\mathrm{Res}_P^G(M))$. More generally, it is true for any reductive group H in good characteristic, as was recently proved by P. Slodowy [14]. His paper also discusses extensions of Weil's method for the proof of the theorem above.

3. G-algebras

Let G be a finite group. A G-**algebra** over F is a finite-dimensional (associative and unitary) algebra A over F, together with a fixed homomorphism $G \to \mathrm{Aut}(A)$. We denote by $\mathrm{J}(A)$ the radical, by $\mathrm{Z}(A)$ the center and by $\mathrm{U}(A)$ the group of units of A. Then G acts on $\mathrm{U}(A)$ and we can form, in case A is commutative, the cohomology groups $\mathrm{H}^i(G, \mathrm{U}(A))$. We have the following finiteness result.

PROPOSITION: *Let G be a finite group, and let A be a commutative G-algebra over an algebraically closed field F whose characteristic does not divide the order of G. Then, for $i \geq 1$, the cohomology group $\mathrm{H}^i(G, \mathrm{U}(A))$ is finite.*

Proof: There is an isomorphism of G-modules $\mathrm{U}(A) \cong \mathrm{U}(A/\mathrm{J}(A)) \times (1 + \mathrm{J}(A))$ so that

$$\mathrm{H}^i(G, \mathrm{U}(A)) \cong \mathrm{H}^i(G, \mathrm{U}(A/\mathrm{J}(A))) \times \mathrm{H}^i(G, 1 + \mathrm{J}(A)).$$

Since the map $1 + \mathrm{J}(A) \to 1 + \mathrm{J}(A)$, $x \mapsto x^{|G|}$, is bijective, we have $\mathrm{H}^i(G, 1 + \mathrm{J}(A)) = 0$. Thus we can replace A by $A/\mathrm{J}(A)$ and therefore assume that A is semisimple. If $A \cong A_1 \times A_2$ with G-algebras A_1, A_2 over F then

$$\mathrm{H}^i(G, \mathrm{U}(A)) \cong \mathrm{H}^i(G, \mathrm{U}(A_1)) \times \mathrm{H}^i(G, \mathrm{U}(A_2)).$$

Hence we may assume that A is a direct product of copies of F transitively permuted under the action of G. Now Shapiro's Lemma implies that $\mathrm{H}^i(G, \mathrm{U}(A)) \cong \mathrm{H}^i(H, \mathrm{U}(F))$ for a subgroup H of G. We have an exact sequence

$$1 \longrightarrow \mu(F) \longrightarrow \mathrm{U}(F) \longrightarrow \mathrm{U}(F)/\mu(F) \longrightarrow 1$$

where $\mu(F)$ denotes the group of all roots of unity in F and $\mathrm{U}(F)/\mu(F)$ is a uniquely divisible group. Hence the long exact sequence in cohomology shows that $\mathrm{H}^i(H, \mathrm{U}(F)) \cong \mathrm{H}^i(H, \mu(F))$, and the latter group is certainly finite. ∎

4. Crossed products

Let G be a finite group. A **G-graded ring** is a ring A, together with a fixed decomposition $A = \bigoplus_{g \in G} A_g$ into additive subgroups A_g satisfying $A_g A_h \subseteq A_{gh}$ for $g, h \in G$. In this case, A_1 is a subring of A containing 1_A. We are interested in classifying all G-graded rings where the identity component A_1 is a given ring R.

There are at least two sensible ways to define isomorphisms between such G-graded rings. We call two G-graded rings $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ with $A_1 = R = B_1$ **weakly equivalent** if there is an isomorphism of rings $\phi: A \to B$ satisfying $\phi(A_g) = B_g$ for $g \in G$. Moreover, we call A and B **equivalent** if ϕ can be chosen to satisfy, in addition, $\phi|A_1 = \mathrm{id}_R$. In the literature (see [7, 12]), usually equivalence classes of G-graded rings are considered. For our purpose, however, weak equivalence classes are more important.

We start by recollecting some of the known facts, at least for the special class of G-graded rings called crossed products. A G-graded ring $A = \bigoplus_{g\in G} A_g$ is called a **crossed product** if $A_g \cap \mathrm{U}(A) \neq \emptyset$ for $g \in G$. In this case, $A_gA_h = A_{gh}$ for $g, h \in G$. From now on, we fix a finite group G and a ring R and consider the class of crossed products $A = \bigoplus_{g\in G} A_g$ of G with R, i.e. with $A_1 = R$.

A **parameter set** of G in R is a pair (α, γ) of maps

$$\alpha: G \longrightarrow \mathrm{Aut}(R), \quad g \longmapsto \alpha_g,$$

and

$$\gamma: G \times G \longrightarrow \mathrm{U}(R), \quad (g,h) \longmapsto \gamma(g,h),$$

satisfying

$$\alpha_g \circ \alpha_h = \iota_{\gamma(g,h)} \circ \alpha_{gh} \quad \text{and} \quad \gamma(g,h)\gamma(gh,k) = \alpha_g(\gamma(h,k))\gamma(g,hk)$$

for $g, h, k \in G$; here ι_r denotes the inner automorphism $R \to R$, $x \mapsto rxr^{-1}$, of R, for $r \in \mathrm{U}(R)$. Two parameter sets (α, γ) and (α', γ') of G in R are called **equivalent** if there are elements $r(g) \in \mathrm{U}(R)$ such that

$$\alpha'_g = \iota_{r(g)} \circ \alpha_g \quad \text{and} \quad \gamma'(g,h) = r(g)\alpha_g(r(h))\gamma(g,h)r(gh)^{-1}$$

for $g, h \in G$. This does indeed define an equivalence relation on the set of all parameter sets of G in R.

Every crossed product $A = \bigoplus_{g\in G} A_g$ of G with R defines an equivalence class of parameter sets of G in R in the following way. We choose $u_g \in A_g \cap \mathrm{U}(A)$ for $g \in G$. Then the map $\alpha_g: R \to R$, $r \mapsto u_g r u_g^{-1}$, is an automorphism of R, and $\gamma(g,h) := u_g u_h u_{gh}^{-1} \in \mathrm{U}(R)$ for $g, h \in G$. Moreover, (α, γ) is a parameter set of G in R. We say that (α, γ) is a parameter set **defined by** A. Note that (α, γ) is not unique since it depends on the choice of u_g for $g \in G$. However, all parameter sets of G in R defined by A form a unique equivalence class.

Also, equivalent crossed products of G with R determine equivalent parameter sets of G in R. Conversely, every parameter set of G in R arises from a crossed product of G with R, unique up to equivalence. Thus:

LEMMA 1: *Equivalence classes of crossed products of G with R are in bijection with equivalence classes of parameter sets of G in R.*

Every parameter set (α, γ) of G in R induces a homomorphism

$$\omega: G \longrightarrow \mathrm{Out}(R) = \mathrm{Aut}(R)/\mathrm{Inn}(R), \quad g \longmapsto \alpha_g \mathrm{Inn}(R).$$

Moreover, ω depends only on the equivalence class of (α, γ). Thus the set of all equivalence classes of parameter sets of G in R splits into disjoint subsets, according to the induced homomorphism $G \to \mathrm{Out}(R)$.

Conversely, however, not every homomorphism $G \to \mathrm{Out}(R)$ is induced by a parameter set of G in R. A necessary and sufficient condition for this can be obtained in the following way. Let $\omega: G \to \mathrm{Out}(R)$ be a homomorphism, and write $\omega(g) = \alpha_g \mathrm{Inn}(R)$ with $\alpha_g \in \mathrm{Aut}(R)$ for $g \in G$. Then the map $G \to \mathrm{Aut}(\mathrm{Z}(R))$, $g \mapsto \alpha_g | \mathrm{Z}(R)$, is a homomorphism and independent of the choice of α_g for $g \in G$. We consider $\mathrm{U}(\mathrm{Z}(R))$ as a G-module via this homomorphism. We can write $\alpha_g \circ \alpha_h = \iota_{r(g,h)} \circ \alpha_{gh}$ with $r(g,h) \in \mathrm{U}(R)$ for $g, h \in G$. Then the map

$$G \times G \times G \longrightarrow \mathrm{U}(\mathrm{Z}(R)), \quad (g,h,k) \longmapsto r(g,h)^{-1} \alpha_g(r(h,k)) r(g,hk) r(gh,k)^{-1},$$

is a 3-cocycle, and the corresponding cohomology class in $\mathrm{H}^3(G, \mathrm{U}(\mathrm{Z}(R)))$ is independent of the choice of α_g and $r(g,h)$ for $g, h \in G$.

LEMMA 2: *The map ω arises from a parameter set of G in R if and only if the corresponding element in $\mathrm{H}^3(G, \mathrm{U}(\mathrm{Z}(R)))$ vanishes.*

Now let $\omega: G \to \mathrm{Out}(R)$ be a homomorphism which is induced by a parameter set, and consider $\mathrm{U}(\mathrm{Z}(R))$ as a G-module via ω. Then the group $\mathrm{Z}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ of 2-cocycles acts on the set of all parameter sets of G in R inducing ω in the following way. For $\zeta \in \mathrm{Z}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ and a parameter set (α, γ) of G in R inducing ω, the parameter set ${}^{\zeta}(\alpha, \gamma) = (\alpha', \gamma')$ is defined by

$$\alpha' := \alpha \quad \text{and} \quad \gamma'(g,h) := \zeta(g,h)\gamma(g,h)$$

for $g, h \in G$. It is easily checked that (α, γ) and ${}^{\zeta}(\alpha, \gamma)$ are equivalent whenever ζ is a 2-coboundary. Thus $\mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ acts on the set of equivalence classes of parameter sets of G in R inducing ω, and in fact $\mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ acts regularly on this set. This shows:

PROPOSITION 1: *The equivalence classes of parameter sets of G in R inducing ω are in bijection (usually not canonical) with $\mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$.*

This means that equivalence classes of parameter sets of G in R can be parametrized by certain pairs (ω, ζ) where $\omega: G \to \mathrm{Out}(R)$ is a homomorphism, $\zeta \in \mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$, and the corresponding action of G on $\mathrm{U}(\mathrm{Z}(R))$ is induced by ω.

Also, the group $\mathrm{Aut}(R)$ acts on the set of all parameter sets (α, γ) of G in R by ${}^{\rho}(\alpha, \gamma) := ({}^{\rho}\alpha, {}^{\rho}\gamma)$ for $\rho \in \mathrm{Aut}(R)$ where ${}^{\rho}\alpha$ and ${}^{\rho}\gamma$ are defined by

$$({}^{\rho}\alpha)_g := \rho \circ \alpha_g \circ \rho^{-1} \quad \text{and} \quad ({}^{\rho}\gamma)(g,h) := \rho(\gamma(g,h))$$

for $g, h \in G$. Moreover, if (α, γ) and (α', γ') are equivalent parameter sets of G in R then so are ${}^{\rho}(\alpha, \gamma)$ and ${}^{\rho}(\alpha', \gamma')$. Thus $\mathrm{Aut}(R)$ acts on the set of equivalence classes of parameter sets of G in R.

It is easy to see that two crossed products $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ of G with R are weakly equivalent if and only if the corresponding equivalence classes of parameter sets of G in R are in the same orbit under $\mathrm{Aut}(R)$. Thus:

PROPOSITION 2: *Weak equivalence classes of crossed products of G with R are in bijection with orbits of* $\mathrm{Aut}(R)$ *on the set of all equivalence classes of parameter sets of G in R.*

If a parameter set (α, γ) of G in R induces the homomorphism $\omega\colon G \to \mathrm{Out}(R)$ then, for $\rho \in \mathrm{Aut}(R)$, the parameter set ${}^{\rho}(\alpha, \gamma)$ induces the homomorphism

$${}^{\rho}\omega\colon G \longrightarrow \mathrm{Out}(R), \quad g \longmapsto \rho\mathrm{Inn}(R) \cdot \omega(g) \cdot \rho^{-1}\mathrm{Inn}(R).$$

Thus, in order to produce a set of representatives for the weak equivalence classes of crossed products of G in R, it suffices to pick one homomorphism $G \to \mathrm{Out}(R)$ out of each conjugacy class under $\mathrm{Out}(R)$, and to compute $\mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ for every such homomorphism.

We now use the main results of sections 2 and 3 to obtain the following finiteness theorem.

THEOREM: *Let F be an algebraically closed field of characteristic p, let R be an algebra of finite dimension over F, and let G be a finite group whose order is not divisible by p. Then there are only finitely many weak equivalence classes of crossed products of G with R.*

Proof: We consider $\mathrm{Out}(R) = \mathrm{Aut}(R)/\mathrm{Inn}(R)$ as a linear algebraic group over F. By the theorem in section 2, there are only finitely many equivalence classes of representations of G in $\mathrm{Out}(R)$. We pick such a representation $\alpha\colon G \to \mathrm{Out}(R)$. Restriction induces a well-defined map res: $\mathrm{Out}(R) = \mathrm{Aut}(R)/\mathrm{Inn}(R) \to \mathrm{Aut}(\mathrm{Z}(R))$, and we consider $\mathrm{Z}(R)$ as a G-algebra via the homomorphism $\mathrm{res} \circ \alpha\colon G \to \mathrm{Aut}(\mathrm{Z}(R))$. Now the proposition in section 3 implies that $\mathrm{H}^2(G, \mathrm{U}(\mathrm{Z}(R)))$ is finite. So the result follows from the discussion above. ∎

5. Donovan's conjecture

In the following, we fix an algebraically closed field F of prime characteristic p and consider p-blocks of a finite group G as subalgebras of the group algebra FG. Our main result is the following one:

THEOREM: *Let p be a prime, and let D be a finite p-group. Then every p-block with defect group D has, up to isomorphism, the form $S \otimes_F T$ where*

(i) *S is a complete matrix algebra over F,*
(ii) *$T = \bigoplus_{x \in X} T_x$ is a crossed product,*
(iii) *X is a finite p'-group whose order divides $|\mathrm{Out}(D)|^2$ where $\mathrm{Out}(D)$ denotes the outer automorphism group of D,*
(iv) *the identity component T_1 of T is a p-block with defect group D, of a finite group H with $H = \langle D^h \colon h \in H \rangle$.*

Proof: Let G be a finite group, and let A be a p-block of FG with defect group D. We wish to prove that A has the form as stated above.

For any normal subgroup N of G, G acts by conjugation on the set of all blocks of FN. It is well-known that the blocks A' of FN satisfying $AA' \neq 0$ (i.e. the blocks of FN **covered** by A) form a single conjugacy class under G (cf. Lemma B in [8]). Let A' be one of these, and denote by $G_{A'}$ the stabilizer of A' in G. Then it is again well-known that there is block A^* of $FG_{A'}$, with defect group conjugate to D, such that A is isomorphic to $\mathrm{Mat}(|G : G_{A'}|, F) \otimes_F A^*$ (see Theorem C in [8]).

This means that it suffices to prove the result for the block A^* of $FG_{A'}$. Repeating this argument as often as possible we see that we may assume that A is **quasi-primitive**, i.e. that, for any normal subgroup N of G, A covers a unique block of FN. Thus let A be quasi-primitive in the following.

We denote by H the subgroup of G generated by all defect groups of A:

$$H := \langle D^g \colon g \in G \rangle.$$

Then H is a normal subgroup of G, and we denote by B the unique block of FH covered by A. It is well-known (cf. Proposition N in [9]) that in this situation every defect group of A is also a defect group of B, so

$$\{D^g: g \in G\} = \{D^h: h \in H\},$$

and $H = \langle D^h: h \in H\rangle$. Since B is G-stable, every element in G acts by conjugation on B. This induces a homomorphism $G \to \mathrm{Aut}(B)$. We denote by K the kernel of the composite homomorphism $G \to \mathrm{Aut}(B) \to \mathrm{Out}(B) = \mathrm{Aut}(B)/\mathrm{Inn}(B)$. Thus K is the normal subgroup of G consisting of all elements in G acting on B by an inner automorphism of B; in particular, $H \subseteq K$. We denote by C the unique block of FK covered by A. Again, C has defect group D.

It is a consequence of (3.5) in [4] that $1_A \in FK$ (see also Corollary 4 in [10]). It follows easily that $1_A = 1_C$; in particular, A is the unique block of FG covering C. It is well-known that this implies that $X := G/K$ is a p'-group (see Theorem 61.5 in [1]). We consider FG as a crossed product of X with FK, as usual. Then $A = 1_C FG$ becomes a crossed product of X with $1_C FK = C$.

Since B and C have a common defect group, Theorem 7 in [10] and its proof imply that there is a simple subalgebra S of $C^H := \{c \in C: hch^{-1} = c \text{ for } h \in H\}$ such that C^H is isomorphic to $S \otimes_F \mathrm{Z}(B)$ and such that C is isomorphic to $S \otimes_F B$ where in both cases the isomorphism is simply given by multiplication. It follows that A is isomorphic to $S \otimes_F \mathrm{C}_A(S)$, again by multiplication, where $T := \mathrm{C}_A(S)$, the centralizer of S in A, is a crossed product of X with $\mathrm{C}_C(S)$, and where $\mathrm{C}_C(S)$ is isomorphic to B.

Thus we have proved all parts of the theorem with the exception of the assertion $|X| \mid |\mathrm{Out}(D)|^2$. For this last part of the proof we use another of the main results of [4]. (Since [4] is a long and technical paper, it may be helpful to the reader to look also at [3] which contains a summary of the main results in [4], including those we need here.)

We denote by b a block of $FD\mathrm{C}_H(D)$ in Brauer correspondence with B. Then b contains a unique irreducible Brauer character ϕ, and we denote by $\mathrm{N}_G(D)_\phi$ the stabilizer of ϕ under the action (by conjugation) of $\mathrm{N}_G(D)$ on Brauer characters of $FD\mathrm{C}_H(D)$. Then, by (12.6) in [4], we have

$$G = \mathrm{N}_G(D)_\phi H \quad \text{and} \quad K = \mathrm{C}_G(D)_\omega H;$$

here $\omega: \mathrm{N}_H(D)_\phi/D\mathrm{C}_H(D) \times \mathrm{C}_G(D)_\phi/\mathrm{C}_H(D) \to \mathrm{U}(F)$ is a certain bilinear map, and $\mathrm{C}_G(D)_\omega$ denotes the group of all elements $x \in \mathrm{C}_G(D)_\phi$ satisfying

$\omega(y, x\mathrm{C}_H(D)) = 1$ for all $y \in \mathrm{N}_H(D)_\phi/D\mathrm{C}_H(D)$. Thus $\mathrm{C}_G(D)_\phi/\mathrm{C}_G(D)_\omega$ is isomorphic to a subgroup of $\mathrm{Hom}(\mathrm{N}_H(D)_\phi/D\mathrm{C}_H(D), \mathrm{U}(F))$. Since $\mathrm{N}_H(D)_\phi/D\mathrm{C}_H(D)$ is naturally isomorphic to a subgroup of $\mathrm{Out}(D)$ we obtain

$$|\mathrm{C}_G(D)_\phi/\mathrm{C}_G(D)_\omega| \,\big|\, |\mathrm{Out}(D)|.$$

Now

$$\begin{aligned}|X| = |G/K| &= |\mathrm{N}_G(D)_\phi H/\mathrm{C}_G(D)_\omega H| \\ &= |\mathrm{N}_G(D)_\phi H/\mathrm{C}_G(D)_\phi H| \cdot |\mathrm{C}_G(D)_\phi H/\mathrm{C}_G(D)_\omega H|\end{aligned}$$

where

$$\begin{aligned}\mathrm{N}_G(D)_\phi H/\mathrm{C}_G(D)_\phi H &\cong \mathrm{N}_G(D)_\phi/\mathrm{N}_G(D)_\phi \cap \mathrm{C}_G(D)_\phi H \\ &= \mathrm{N}_G(D)_\phi/\mathrm{C}_G(D)_\phi \mathrm{N}_H(D)_\phi\end{aligned}$$

is isomorphic to a factor group of $\mathrm{N}_G(D)_\phi/D\mathrm{C}_G(D)_\phi$ which, in turn, is isomorphic to a subgroup of $\mathrm{Out}(D)$, and where

$$\begin{aligned}\mathrm{C}_G(D)_\phi H/\mathrm{C}_G(D)_\omega H &\cong \mathrm{C}_G(D)_\phi/\mathrm{C}_G(D)_\phi \cap \mathrm{C}_G(D)_\omega H \\ &= \mathrm{C}_G(D)_\phi/\mathrm{C}_G(D)_\omega \mathrm{C}_H(D) \\ &= \mathrm{C}_G(D)_\phi/\mathrm{C}_G(D)_\omega.\end{aligned}$$

Thus $|X| \,\big|\, |\mathrm{Out}(D)|^2$ as we wished to prove. ∎

With notation as above, $S \otimes_F T$ is Morita equivalent to T. The basic subalgebra of T_1 has, of course, the form eT_1e where e is an idempotent in T_1, unique up to conjugation with units in T_1. It follows easily that $Y := eTe$ is a crossed product of X with eT_1e. Moreover, eTe is Morita equivalent to T since

$$TeT = TT_1eT_1T = TT_1T = T.$$

Hence we obtain:

COROLLARY: *Let p be a prime, and let D be a finite p-group. Then every p-block with defect group D is Morita equivalent to a crossed product $Y = \bigoplus_{x \in X} Y_x$ satisfying the following:*

(i) *X is a finite p'-group whose order divides $|\mathrm{Out}(D)|^2$,*

(ii) *the identity component Y_1 of Y is a basic subalgebra of a block with defect group D, in a finite group H with $H = \langle D^h \colon h \in H \rangle$.*

The important point is, of course, that, for a given D, there are only finitely many possibilities for X. Moreover, the main result in section 4 implies that, given a finite p'-group X and a finite-dimensional F-algebra Y_1 there are, up to isomorphism, only finitely many crossed products Y with identity component Y_1. Thus it suffices to show that there are only finitely many possibilities for Y_1. Since two F-algebras are Morita equivalent if and only if their basic subalgebras are isomorphic we conclude:

PROPOSITION: *In order to prove Donovan's conjecture, it suffices to show the following:*

For any prime p and any finite p-group D, there are only finitely many Morita equivalence classes of p-blocks with defect group D, in finite groups H with $H = \langle D^h \colon h \in H \rangle$.

If the question in section 2 could be answered positively, one could perhaps deduce stronger reductions for Donovan's conjecture.

Let b be a p-block with defect group D in a finite group H. If D is a Sylow p-subgroup of H (e.g. if b is the principal p-block of H) then $\langle D^h \colon h \in H \rangle = \mathrm{O}^{p'}(H)$. Hence, in the general case, the normal subgroup $\langle D^h \colon h \in H \rangle$ of H can be considered as an analogue, for p-blocks, of $\mathrm{O}^{p'}(H)$. We therefore propose to denote this subgroup by $\mathrm{O}^{b'}(H)$. The condition on H in the proposition then reads as $H = \mathrm{O}^{b'}(H)$.

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